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The Cayley–Hamilton theorem and inverse problems for multiparameter systems

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Abstract

We review some of the current research in multiparameter spectral theory. We prove a version of the Cayley–Hamilton theorem for multiparameter systems and list a few inverse problems for such systems. Some consequences of results on determinantal representations proved by Dixon, Dickson and Vinnikov for the inverse problems are discussed.

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1. Introduction

If $w(\mathbf{x})$ is a homogeneous polynomial of degree d in the polynomial algebra $F[\mathbf{x}] = F[x_0, x_1, \ldots, x_n]$ do there exist matrices A_j, such that

$$\det\left(\sum_{j=0}^{n} A_j x_j\right) = w(\mathbf{x})?$$

Dixon [9] and Dickson [8] were the first to consider the question. Dixon showed that the answer is positive for all $w(\mathbf{x})$ if $n = 2$ and Dickson showed that the answer is generically no if $n \geqslant 3$ and $d \geqslant 2$ except when $n = 3$ with $d = 2$ or 3 and $n = 4$ with

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$d = 2$. The case $n = 2$ was studied in detail by Vinnikov [22,23] (see also Fiedler's paper [12]). The cases $n \geqslant 3$ have not yet been studied in detail. An exception is work by Giacobazzi [13] on some special cases of $n = 3$. In addition, there is a '*curious fact*' known in the theory of generalized Clifford algebras and linear maximal Cohen–Macaulay modules that for each homogeneous polynomial $w(\mathbf{x})$ there is a number $k \in \mathbb{N}$ such that the answer is positive for the power $w(\mathbf{x})^k$ (see [4,17]).

Here we propose a more general question. Given a regular sequence $w_1, w_2, \ldots, w_n$ of homogeneous polynomials in $F[\mathbf{x}]$ when do there exist matrices A_{ij} such that

$$\det\left(\sum_{j=0}^{n} A_{ij}x_j\right) = w_i(\mathbf{x}), \quad i = 1, 2, \ldots, n?$$

The motivation to propose this problem comes from multiparameter spectral theory [3]. A recent attempt [6] to answer the conjecture of Faierman [11, pp. 122–123] led us to study the inverse problem proposed above for $n = 2$. Faierman works with coupled boundary value problems for ordinary differential equations of Sturm–Liouville type and so the linear maps A_{ij} in his setup act on infinite dimensional vector spaces. Although we have not been successful in resolving the conjecture it might be interesting for a wider readership to present finite-dimensional results. In particular, we hope that techniques of [3,5,6] might help in 'translating' algebraic results into the setup of differential equations. We discuss details later in the paper.

Let us conclude the introduction with a brief overview of the paper. In the second section we introduce Faierman's setup and explain the motivation for our discussion. In Section 3 we introduce Atkinson's abstract algebraic setup and in Section 4 we rephrase it in the language of commutative algebra. In the process we prove a version of the Cayley–Hamilton theorem for multiparameter systems. In Section 4 we introduce several forms of the inverse problem for multiparameter systems and discuss some consequences of results of Dickson [8] and Vinnikov [22] for these problems.

2. Motivation

Faierman [11] considers a two-parameter eigenvalue problem involving a class of coupled Sturm–Liouville boundary value problems

$$\frac{\mathrm{d}}{\mathrm{d}t_j}\left(p_j(t_j)\frac{\mathrm{d}y_j}{\mathrm{d}t_j}\right) + \big((-1)^{j+1}\mu_1 a_{j1}(t_j) + (-1)^j\mu_2 a_{j2}(t_j) - q_j(t_j)\big)y_j = 0, \tag{1}$$

$$y_j(0)\cos\alpha_j - p_j(0)\frac{\mathrm{d}y_j}{\mathrm{d}t_j}(0)\sin\alpha_j = 0, \quad 0 \leqslant \alpha_j < \pi, \tag{2}$$

$$y_j(1)\cos\beta_j - p_j(1)\frac{\mathrm{d}y_j}{\mathrm{d}t_j}(1)\sin\beta_j = 0, \quad 0 < \beta_j \leqslant \pi, \tag{3}$$

for $j = 1, 2,$ where $\boldsymbol{\mu} = (\mu_1, \mu_2)$ are parameters and $t_j \in \mathscr{I} := [0,\ 1]$. Following [11, pp. 2, 10] we assume that:

(i) for $j = 1, 2$ the functions $p_j, q_j, a_{jk},\ k = 1, 2$ are real valued, p_j, a_{jk} are Lipschitz continuous, p_j is positive and q_j is essentially bounded,
(ii) the function $\omega(t_1, t_2) = a_{11}(t_1)a_{22}(t_2) - a_{12}(t_1)a_{21}(t_2)$ on $\mathscr{I}^2$ is not identically 0,
(iii) $a_{j1}(t_j) > 0$ for $t_j \in \mathscr{I}$ and $j = 1, 2$.

It is easily seen that $A_{j0}y_j = -y_j'' + q_j y_j$ is a self-adjoint operator with domain $\mathscr{D} = \{y;\ \ y \text{ is } C^1 \text{ on } \mathscr{I},\ y' \text{ is absolutely continuous on } \mathscr{I},\ y'' \in L_2(\mathscr{I}),\ \ y(0) = y(1) = 0\}$. We denote by A_{jk} the operator of multiplication by a_{jk}. Then we associate with (1)–(3) an abstract two-parameter system

$$A_{j0}y_j = \lambda A_{j1}y_j + (-1)^j \mu A_{j2}y_j, \quad j = 1, 2, \tag{4}$$

for $y_j \in \mathscr{D}$. There is an infinite number of nonzero eigenvalues of (1), each with finite multiplicity (see for example [3,5,11,21,24]). In general, there is possibly a finite number of nonreal eigenvalues and a finite number of nonsemisimple eigenvalues [11,24].

The above two-parameter Sturm–Liouville example shows that even for self-adjoint cases there is need for a general theory that will include understanding of the behavior of root subspaces (i.e. subspaces spanned by eigenfunctions and generalized eigenfunction) at nonsemisimple eigenvalues. Faierman [11, Conjecture 6.1, p. 122] conjectured a precise form of generalized eigenfunctions at nonreal nonsemisimple eigenvalues. It is not known if the conjecture holds. In [6] we give a different basis of eigenfunctions and generalized eigenfunction. Bases of eigenfunctions and generalized eigenfunctions are used to get a Fourier type expansion of the general solution of (1)–(3).

3. Abstract setup

The attempts to formulate an abstract setup to study two and more parameter systems go back at least to Carmichael [7]. It was Atkinson (see [1–3]) who introduced the abstract setup which has been studied since by a number of mathematicians (see for example books and lecture notes [11,18,19,21,24]). We give a brief introduction to Atkinson's setup. For details we refer to [3]. To avoid a number of technical difficulties we assume that the underlying vector spaces are finite-dimensional. In the infinite-dimensional setup various additional assumptions are made (see [5]). These are natural when treating Sturm–Liouville boundary value problems. It is our experience that, under the uniform ellipticity condition (iii) or under the Fredholmness condition on eigenvalues [6, Assumption III], the bases construction in the finite-dimensional situation can be transferred to the infinite dimensional situation. Therefore there is hope that answers to our problems in finite dimensions can be of use when treating Faierman's construction in [11, Conjecture 6.1, p. 122].

Consider an *n-parameter system* ($n \geqslant 2$) of the form

$$W_j(\mathbf{x}) = \sum_{k=0}^{n} A_{jk} x_k, \quad j = 1, 2, \ldots, n, \tag{5}$$

where A_{jk} ($k = 0, 1, 2, \ldots, n$) are linear maps acting on a finite-dimensional vector space V_j ($j = 1, 2, \ldots, n$) over an algebraically closed field F (in applications $F = \mathbb{C}$) and $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ are variables (also called parameters).

The linear maps A_{jk}, $k = 0, 1, 2, \ldots, n$, induce linear maps $A_{jk}^{\dagger}$ on the tensor product $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by means of

$$A_{jk}^{\dagger}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_{j-1} \otimes A_{jk} v_j \otimes v_{j+1} \otimes \cdots \otimes v_n$$

on decomposable tensors, extended by linearity and continuity to the whole of V. The operator Δ_0 on V is defined by

$$\Delta_0 = \det \left[A_{jk}^{\dagger}\right]_{j,k=1}^{n} \tag{6}$$

and operators Δ_k ($k = 1, 2, \ldots, n$) are obtained by replacing the kth column in (6) by $[-A_{j0}^{\dagger}]_{j=1}^{n}$.

We say that a multiparameter system (5) is *regular* if there are scalars α_j, $j = 0, 1, \ldots, n$ such that the linear map $\sum_{j=0}^{n} \alpha_j \Delta_j : V \to V$ is invertible.

Henceforth, we consider only regular multiparameter systems. Without loss we also assume that the operator Δ_0 is invertible. This can be achieved by an invertible linear substitution of variables $\boldsymbol{x}$.

We define operators $\Gamma_j : V \to V$ by $\Gamma_j = \Delta_0^{-1} \Delta_j$, $j = 0, 1, \ldots, n$.

Theorem 1 [3, Theorems 6.7.1 and 6.7.2]. *The operators* Γ_j *commute and*

$$\sum_{k=0}^{n} A_{jk}^{\dagger} \Gamma_k = 0. \tag{7}$$

4. Connections with commutative algebra

We write $w_j(\mathbf{x}) = \det W_j(\mathbf{x})$ and $\boldsymbol{\Gamma} = (\Gamma_0, \Gamma_1, \ldots, \Gamma_n)$. The polynomials w_j, $j = 1, 2, \ldots, n$ are elements of the polynomial ring $R = F[\mathbf{x}]$. We denote the multiindex $(k_0, k_1, \ldots, k_n)$ by $\mathbf{k}$ and write $\mathbf{x}^{\mathbf{k}} = x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n}$. We denote by $\mathbf{e}_r$ the multiindex $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 on the rth place.

The following is a generalization of the Cayley–Hamilton theorem.

Theorem 2. *If* w_j *and* $\boldsymbol{\Gamma}$ *are as above then*

$$w_j(\boldsymbol{\Gamma}) = 0$$

for $j = 1, 2, \ldots, n$.

Proof. Suppose that adj $W_j(\mathbf{x}) = \sum_{\mathbf{k}} B_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$ is the adjoint matrix of $W_j(\mathbf{x})$. Here the summation is over all the multiindices $\mathbf{k} = (k_0, k_1, \ldots, k_n)$ such that $\sum_{l=0}^{n} k_l = \dim V_j - 1$. Then it follows that

$$(\operatorname{adj} W_j(\mathbf{x}))^{\dagger} W_j(\mathbf{x})^{\dagger} = w_j(\mathbf{x})I,$$

where I is the identity map on V. We see that

$$w_j(\mathbf{x})I = \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger}\left(\sum_{r=0}^{n} A_{jr}^{\dagger} x_r\right)\mathbf{x}^{\mathbf{k}} = \sum_{\mathbf{l}}\left(\sum_{r} B_{\mathbf{l}-\mathbf{e}_r} A_{jr}\right)^{\dagger}\mathbf{x}^{\mathbf{l}},$$

where the latter summation is over all r such that $l_r \geqslant 1$. Note also that the matrix in the brackets is a scalar matrix. Now Cramer's rule (7) implies that

$$w_j(\boldsymbol{\Gamma}) = \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger}\left(\sum_{r=0}^{n} A_{jr}^{\dagger}\Gamma_r\right)\boldsymbol{\Gamma}^{\mathbf{k}} = 0. \qquad \square$$

Next we define the notions of eigenvalues and spectra. It is customary in multiparameter spectral theory to consider the nonhomogeneous situation. Therefore, we set $x_0 = 1$. In the rest of the paper we write $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{\Gamma} = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, etc., unless stated otherwise.

An n-tuple $\boldsymbol{\alpha} \in F^n$ is called an *eigenvalue of a multiparameter system* (5) if all $W_i(\boldsymbol{\alpha})$ are singular. The set of all eigenvalues is called the *spectrum* of (5) and denoted by $\sigma(\mathbf{W})$. An eigenvalue $\boldsymbol{\alpha} \in \sigma(\mathbf{W})$ is called *geometrically simple* if $\dim\ker W_i(\boldsymbol{\alpha}) = 1$ for $i = 1, 2, \ldots, n$.

An n-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in F^n$ is called an *eigenvalue of the system of commuting linear transformations* $\boldsymbol{\Gamma}$ if

$$\mathscr{M}_{\boldsymbol{\alpha}} = \bigcap_{i=1}^{n} \ker(\Gamma_i - \alpha_i I) \neq \{0\}.$$

The set of all these eigenvalues is called the *spectrum* of $\boldsymbol{\Gamma}$ and denoted by $\sigma(\boldsymbol{\Gamma})$. The subspace $\mathscr{M}_{\boldsymbol{\alpha}}$ is called the *eigenspace* of $\mathbf{W}$ at $\boldsymbol{\alpha}$.

Theorem 3 [3, Theorem 6.9.1]. *The spectrum $\sigma(\mathbf{W})$ of a multiparameter system $\mathbf{W}$ and the spectrum $\sigma(\boldsymbol{\Gamma})$ of its associated system $\boldsymbol{\Gamma}$ coincide. For a given eigenvalue $\boldsymbol{\alpha}$ we have that*

$$\mathscr{M}_{\boldsymbol{\alpha}} = \ker W_1(\boldsymbol{\alpha}) \otimes \ker W_2(\boldsymbol{\alpha}) \otimes \cdots \otimes \ker W_n(\boldsymbol{\alpha}). \tag{8}$$

Theorem 4 [3, 16, Theorem 5.1]. *The following are equivalent*:

(1) *multiparameter system* (5) *is regular*,
(2) *the spectrum $\sigma(\mathbf{W})$ is finite*,
(3) *polynomials $w_1, w_2, \ldots, w_n$ form a regular sequence in R.*

We denote by I the ideal in R generated by the polynomials w_j, $j = 1, 2, \ldots, n$. Then the quotient $A = R/I$ is an artinian algebra. Since w_j, $j = 1, 2, \ldots, n$ form a regular sequence it is a complete intersection, and therefore also a Gorenstein and a Cohen–Macaulay algebra (see e.g. [10] for definitions). It is well known that an artinian and noetherian algebra is a direct product of local artinian algebras. Then $A = \prod_{\alpha\in\sigma(\mathbf{W})} \mathscr{A}_\alpha$, where $\mathscr{A}_\alpha$ is the localization of A at the maximal ideal generated by the polynomials $x_j - \alpha_j$, $j = 1, 2, \ldots, n$. Consult [10] for details. Note that $\mathscr{A}_\alpha \cong R_\alpha/I_\alpha$. We will often omit the subscript and write $\mathscr{A} = \mathscr{A}_\alpha$.

Each $W_j(\mathbf{x})$ induces an R-module map of the free module $R \otimes V_j$. Since $\det W_j(\mathbf{x}) = w_j(\mathbf{x})$ is nonzero this map is injective. We denote by M_j its cokernel. In a similar way, we denote by $\mathscr{M}_j(\boldsymbol{\alpha})$ the cokernel of the $\mathscr{A}_\alpha$-module map induced by $W_j(\mathbf{x})$ on the free module $\mathscr{A}_\alpha \otimes V_j$. Each map $\Gamma_j - x_j$ induces a module map of the free R-module $R \otimes V$ and of the free $\mathscr{A}$-module $\mathscr{A} \otimes V$. We denote by M the cokernel of the module map $[\Gamma_j - x_j]_{j=1}^n : (R \otimes V)^n \to R \otimes V$ and by $\mathscr{M}$ the cokernel of the map $[\Gamma_j - x_j]_{j=1}^n : (\mathscr{A} \otimes V)^n \to \mathscr{A} \otimes V$. Then

$$M \cong M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n \quad \text{and} \quad \mathscr{M} \cong \mathscr{M}_1 \otimes_{\mathscr{A}} \mathscr{M}_2 \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \mathscr{M}_n.$$

See [16] for details. The discussion of this paragraph can also be set in the language of coalgebras [14–16].

Theorem 5. *If $\boldsymbol{\alpha}$ is a geometrically simple eigenvalue then $\mathscr{M}$ is a free $\mathscr{A}$-module of rank one. The algebra $\mathscr{B}$ generated by the restrictions $\Gamma_j|_{\mathscr{M}}$, $j = 1, 2, \ldots, n$, is isomorphic to $\mathscr{A}$.*

Proof. The first part of the theorem is proved in [16, Corollary 3.1]. By Theorem 2 the restrictions $\Gamma_j|_{\mathscr{M}}$ satisfy the relations defining the ideal I. Thus the algebra $\mathscr{B}$ that they generate is a quotient of $\mathscr{A}$. But by [20, Corollary 3] the dimensions of $\mathscr{B}$ and $\mathscr{M} \cong \mathscr{A}$ (as vector spaces over F) are equal. Therefore, $\mathscr{B} \cong \mathscr{A}$. □

It is important to study geometrically simple eigenvalues because of the application to the Sturm–Liouville problems. The eigenvalues of a multiparameter system that is induced by a Sturm–Liouville boundary value problem are all geometrically simple.

5. Inverse problems

The inverse problem for multiparameter systems is a problem of existence of multiparameter systems that satisfy some given data. We discuss the following problems:

(1) Given a regular sequence $(w_1, w_2, \ldots, w_n)$ in $F[\mathbf{x}]$ is there a multiparameter system (5) such that $\det W_j(\mathbf{x}) = w_j(\mathbf{x})$ for $j = 1, 2, \ldots, n$?

(2) Given an ideal I in $F[\mathbf{x}]$ such that the quotient $A = F[\mathbf{x}]/I$ is an artinian complete intersection is there a multiparameter system (5) such that $\det W_j(\mathbf{x}) = w_j(\mathbf{x})$, $j = 1, 2, \ldots, n$ is a regular sequence of generators for I?
(3) Given an ideal I in $F[\mathbf{x}]$ such that the quotient $A = F[\mathbf{x}]/I$ is an artinian complete intersection and a faithful A-module M is there a multiparameter system (5) such that $\det W_j(\mathbf{x}) = w_j(\mathbf{x})$, $j = 1, 2, \ldots, n$ is a regular sequence of generators for I and $V \cong M$ as A-modules?
(4) Given an n-tuple $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ of commuting matrices is there a multiparameter system (5) such that the given matrices form its associated system of commuting matrices?
(5) Given a local artinian complete intersection $\mathscr{A}$ is there a Sturm–Liouville problem (1) with an eigenvalue $\boldsymbol{\alpha}$ such that $\mathscr{A} \cong \mathscr{M}_{\boldsymbol{\alpha}}$ as $\mathscr{A}$-modules.

In all the problems we may also, in addition to the existence of solutions, ask for the parameterization of all solutions, when there are any. In the first four problems, one can also ask for the existence of a symmetric (or self-adjoint, if $F = \mathbb{C}$) multiparameter system. This might be of interest when trying to answer the last problem. Partial answers to Problems (1)–(4) follow from results on determinantal representations proved by Dixon [9], Dickson [8] and Vinnikov [22,23].

The Problems (1)–(5) are set for not necessarily homogeneous polynomials, i.e. geometrically they are set in the affine situation. In order to discuss them in the light of the above mentioned results on determinantal representations we have to homogenize all the polynomials. That is, we substitute x_i by x_i/x_0 and multiply a polynomial of degree d by x_0^k for $k \geqslant d$. Since we get a polynomial for each $k \geqslant d$ this might give some additional freedom in the search for answers to our problems.

We denote by $F_d[\mathbf{x}]$ the set of all homogeneous polynomials of degree d together with the polynomial 0. Here $\mathbf{x} = (x_0, x_1, \ldots, x)$. A *determinantal representation* of a polynomial $w(\mathbf{x}) \in F_d[\mathbf{x}]$ is a linear matrix polynomial $W(\mathbf{x}) = \sum_{j=0}^{n} x_j A_j$ such that $w(\mathbf{x}) = \det W(\mathbf{x})$. Dickson [8, Theorem 2] proved that a general polynomial $w \in F_d[x_0, x_1, \ldots, x_n]$ has a determinantal representation if and only if either

(1) $n = 2$,
(2) $n = 3, 4$ and $d = 2$,
(3) $n = d = 3$.

In addition, he proved that for $n = 2$, and $n = 3, 4$ with $d = 2$ every polynomial $w \in F_d[x_0, x_1, \ldots, x_n]$ has a determinantal representation. If none of the conditions (1)–(3) on n and d hold then a general polynomial in $F_d[x_0, x_1, \ldots, x_n]$ does not have a determinantal representation. Using Dickson's result we have the following partial answer to Problem 1. Here we denote by d the largest of the degrees of the polynomials $w_j(\mathbf{x})$.

Theorem 6. *Let F be any field and assume that $(w_1(\mathbf{x}), w_2(\mathbf{x}), \ldots, w_n(\mathbf{x}))$ is a regular sequence in $F[\mathbf{x}]$. If either $n = 2$, or $n = 3, 4$, $d = 2$, then there exists* *a multiparameter system* $W_j(\mathbf{x})$, $j = 1, 2, \ldots, n$ *such that* $w_j(\mathbf{x}) = \det W_j(\mathbf{x})$ *for all* j.

If $n = 2$ and F is an algebraically closed field then a parameterization of all determinantal representations of a nonsingular irreducible polynomial $w \in F[\mathbf{x}]$ is given in Vinnikov's paper [22, p. 129]. Namely, the determinantal representations are parameterized by points on the Jacobian variety of the curve $w(\mathbf{x}) = 0$ that are not on the exceptional subvariety.

Some results are known about self-adjoint and symmetric determinantal representations. Vinnikov [23, Corollary 3.2] proved that each real nonsingular curve $w(x_0, x_1, x_2) = 0$ has a self-adjoint (over $\mathbb{C}$) determinantal representation. He also conjectured in [22, p. 134] that a nonsingular irreducible curve has a symmetric determinantal representation.

Theorem 6 gives a partial answer to Problem 2 as well. If $n = 2$, or $n = 3, 4$ and $m^3 \subset I \subset m^2$ then there is a multiparameter system W_j, $j = 1, 2, \ldots, n$, such that the determinants $\det W_j$ are a regular sequence of generators for I. (Here m is the ideal in R generated by the variables $x_1, x_2, \ldots, x_n$.) The answer to the remaining cases depends on the answer to the following problem:

(6) If I is an ideal $F[\mathbf{x}]$ such that the quotient $F[\mathbf{x}]/I$ is an artinian complete intersection is there a regular sequence $(w_1, w_2, \ldots, w_n)$ of generators for I that all have determinantal representations?

We do not know an answer to Problem 3 if M is arbitrary. If $M \cong \mathscr{A}$ then Theorem 6 gives a partial answer.

If $\mathbf{\Gamma}$ is a geometrically simple n-tuple of commuting maps then the answer to Problem 4 is no unless the algebra $\mathscr{B}$ generated by $\mathbf{\Gamma}$ in $M_n(F)$ is a complete intersection. If so then Problem 4 is equivalent to the Problem 3 with $\mathscr{A} = \mathscr{B}$ and $M = F^n$. If $\mathbf{\Gamma}$ is not geometrically simple then no answer to Problem 4 is known. Then $\mathscr{B}$ is a proper quotient of $\mathscr{A}$.

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